<u>Product theorem for limits</u> Let $X$ be a metric space.

Let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be functions and let $a \in X$. Assume that

$$\lim_{x\to a} f(x) \text{ exists and } \lim_{x\to a} g(x) \text{ exists.}$$

Then

$$\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right).$$

<u>Proof</u> Let $l_1 = \lim_{x\to a} f(x)$ and $l_2 = \lim_{x\to a} g(x)$.

To show: $\lim_{x\to a} f(x)g(x) = l_1 l_2$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $|x-a|<\delta$ then $|f(x)g(x) - l_1 l_2| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

Let $\varepsilon_1 = \min\left(\frac{\varepsilon}{|l_1|+|l_2|+1}, 1\right)$

Since $\lim_{x\to a} f(x) = l_1$, there exists $\delta_1 \in \mathbb{R}_{>0}$ such that if $|x-a|<\delta_1$ then $|f(x) - l_1| < \varepsilon_1$

Since $\lim_{x\to a} g(x) = l_2$, there exists $\delta_2 \in \mathbb{R}_{>0}$ such that if $|x-a|<\delta_2$ then $|g(x) - l_2| < \varepsilon_1$

Let $\delta = \min(\delta_1, \delta_2)$. Then

$$|f(x)g(x) - l_1 l_2| = |(f(x)-l_1)g(x) + l_1(g(x)-l_2)|$$

$$\leq |(f(x)-l_1)g(x)| + |l_1(g(x)-l_2)|,$$ by the triangle inequality

$$= |(f(x)-l_1)(g(x)-l_2) + (f(x)-l_1)l_2|$$
$$+ |l_1||g(x)-l_2|$$

$$\leq |(f(x)-l_1)(g(x)-l_2)| + |(f(x)-l_1)(l_2)| + |l_1||g(x)-l_2|$$

$$= |f(x)-l_1| \cdot |g(x)-l_2| + |f(x)-l_1| \cdot |l_2| + |l_1||g(x)-l_2|$$

$$< \varepsilon_1^2 + \varepsilon_1|l_2| + |l_1|\varepsilon_1 = \varepsilon_1(|l_1|+|l_2|+\varepsilon_1)$$

$$\leq \varepsilon_1(|l_1|+|l_2|+1)$$

$$\leq \varepsilon.$$

~~So~~ $\cancel{\lim_{x \to a} (f(x)g(x) = l_1 l_2)}$

So $\lim_{x \to a} f(x)g(x) = l_1 l_2$

(1) (a) A <u>metric space</u> is a set $X$ with a function

$d: X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $p \in X$ then $d(p,p)=0$,

(b) If $p,q \in X$ and $p \neq q$ then $d(p,q) \neq 0$,

(c) If $p,q \in X$ then $d(p,q)=d(q,p)$,

(d) If $p,q,r \in X$ then $d(p,r) \leq d(p,q)+d(q,r)$.

(b) Let $X$ be a metric space

The metric space $X$ is <u>complete</u> if it satisfies

if $(a_n)$ is a sequence in $X$ and $(a_n)$ is Cauchy

then $(a_n)$ converges in $X$.

(c) Let $X$ be a metric space.

The <u>completion</u> of $X$ is the smallest metric

space $\hat{X}$ such that $\hat{X} \supseteq X$ and $\hat{X}$ is complete.

Examples: $\hat{X}=\mathbb{R}$ with the metric given by

$d(p,q)=|q-p|$ is a metric space.

$X=\mathbb{Q}$ with the metric given by

$d(p,q)=|q-p|$ is a metric space.

From $(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x+\frac{\frac{1}{2}(\frac{-1}{2})}{2!}x^2+\frac{\frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})}{3!}x^3+\cdots$

we get a sequence $(s_1, s_2, \ldots)$ given by

$s_1 = 1, \quad s_2 = 1 + \frac{1}{2}, \quad s_3 = 1 + \frac{1}{2} + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!},$

$$s_4 = 1 + \frac{1}{2} + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!} + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{3}{2}\right)}{3!},$$

$$s_5 = 1 + \frac{1}{2} + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!} + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{3}{2}\right)}{3!} + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{-5}{2}\right)}{4!}; \quad \ldots$$

The sequence $(s_1, s_2, s_3, \ldots)$ in $\mathbb{Q}$ is a Cauchy sequence that does not converge.

The sequence $(s_1, s_2, s_3, \ldots)$ in $\mathbb{R}$ is a Cauchy sequence that converges to $\sqrt{2}$.

$\mathbb{R}$ is the completion of $\mathbb{Q}$.

(2) Let $(a_n)$ be a sequence in a metric space $X$.

Show that if $(a_n)$ converges then $(a_n)$ is Cauchy.

Proof: Assume that $(a_n)$ converges.

Let $l \in X$ such that if $\varepsilon_1 \in \mathbb{R}_{>0}$ then there exists $N_1 \in \mathbb{Z}_{>0}$ such that if $n_1 \in \mathbb{Z}_{>0}$ and $n_1 > N_1$ then $d(a_{n_1}, l) < \varepsilon_1$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(a_m, a_n) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let $\varepsilon_1 = \varepsilon/2$. Then there exists $N_1 \in \mathbb{Z}_{>0}$ such that if $n_1 > N_1$ then $d(a_{n_1}, l) < \varepsilon_1$.

~~Let $N = N_1$~~

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(a_m, a_n) < \varepsilon$.

Let $N = N_1$

To show: If $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(a_m, a_n) < \varepsilon$.

Assume $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$.

So $m > N_1$ and $n > N_1$ and $d(a_m, l) < \varepsilon_1$ and $d(a_n, l) < \varepsilon_1$.

To show: $d(a_m, a_n) < \varepsilon$.

$$d(a_m, a_n) \leq d(a_m, \ell) + d(a_n, \ell),$$ by the triangle inequality,

$$< \varepsilon_1 + \varepsilon_1 = \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So $d(a_m, a_n) < \varepsilon$.

So $(a_n)$ is Cauchy. //

(3) Let $(a_n)$ be a sequence in $\mathbb{R}$. (Note that there is a misprint here — otherwise the question is too easy). Show that if $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Proof Assume $\sum_{n=1}^{\infty} |a_n|$ converges

Let
$$A_1 = |a_1|,\ A_2 = |a_1| + |a_2|,\ A_3 = |a_1| + |a_2| + |a_3|, \dots$$

Then $(A_1, A_2, A_3, \dots)$ converges.

So $(A_1, A_2, \dots)$ is Cauchy.

Let $s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3, \dots$

To show: $(s_1, s_2, s_3, \dots)$ converges.

Since $(s_1, s_2, s_3, \dots)$ is a sequence in $\mathbb{R}$, $(s_1, s_2, \dots)$ will converge if $(s_1, s_2, s_3, \dots)$ is Cauchy.

To show: $(s_1, s_2, s_3, \dots)$ is Cauchy.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $|s_m - s_n| < \varepsilon$

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Since $(A_1, A_2, \ldots)$ is Cauchy there exists an $N_1 \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N_1$ and $n > N_1$ then $|A_m - A_n| < \varepsilon$.

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $m \in \mathbb{Z}_{>0}, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $|s_m - s_n| < \varepsilon$.

Let $N = N_1$

To show: If $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $|s_m - s_n| < \varepsilon$.

Assume $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$.

Since $m > N_1$ and $n > N_1$ then $|A_m - A_n| < \varepsilon$.

To show: $|s_m - s_n| < \varepsilon$.

If $m > n$ then

$$
\begin{aligned}
|s_m - s_n| &= |a_{n+1} + a_{n+2} + \cdots + a_m| \\
&\le |a_{n+1}| + |a_{n+2}| + \cdots + |a_m|, \quad \text{by the triangle inequality,} \\
&= A_m - A_n = |A_m - A_n| < \varepsilon.
\end{aligned}
$$

If $n > m$ then

$$
\begin{aligned}
|s_n - s_m| &= |a_{m+1} + a_{m+2} + \cdots + a_n| \\
&\le |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| \\
&= A_n - A_m = |A_n - A_m| < \varepsilon.
\end{aligned}
$$

So $(s_1, s_2, \ldots)$ is a Cauchy sequence in $\mathbb{R}$.

So $(s_1, s_2, \ldots)$ converges.

So $\sum_{n=1}^{\infty} a_n$ converges. $\square$

(4) $\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) - 1}{x^2}$

$= \lim_{x\to 0} \frac{-\frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots}{x^2} = \lim_{x\to 0} -\frac{1}{2} + \frac{1}{4!}x^2 - \frac{1}{6!}x^6 + \cdots$

Let $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that if $|x| < \delta$

then $\left| -\frac{1}{2} + \frac{1}{4!}x^2 - \frac{1}{6!}x^6 + \cdots - \left(-\frac{1}{2}\right)\right| < \varepsilon$.

Let $\delta = \min\left(\sqrt{\frac{\varepsilon}{2}}, \sqrt{\frac{1}{2}}\right)$.

To show: If $|x| < \delta$ then $\left| -\frac{1}{2} + \frac{1}{4!}x^2 - \frac{1}{6!}x^6 + \cdots - \left(-\frac{1}{2}\right)\right| < \varepsilon$.

Assume $|x| < \delta$.

Then

$\left| -\frac{1}{2} + \frac{1}{4!}x^2 - \frac{1}{6!}x^6 + \cdots - \left(-\frac{1}{2}\right)\right| = \left|\frac{1}{4!}x^2 - \frac{1}{6!}x^6 + \cdots\right|$

$= \left|x^2\left(\frac{1}{4!} + \frac{-1}{6!}x^2 + \frac{1}{8!}x^4 + \cdots\right)\right|$

$\leq |x|^2\left(\left|\frac{1}{4!}\right| + \left|\frac{-1}{6!}x^2\right| + \left|\frac{1}{8!}x^4\right| + \cdots\right)$

$\leq |x|^2\left(1 + |x|^2 + |x|^4 + \cdots\right) \leq |x|^2\left(\frac{1}{1-|x|^2}\right)$

$\leq \delta^2\left(\frac{1}{1-\frac{1}{2}}\right) = 2\delta^2 < \varepsilon.$

So $\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \frac{-1}{2}$.

Then

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} x \cdot \frac{\cos x - 1}{x^2}$$

$$= \left(\lim_{x \to 0} x\right) \cdot \left(\lim_{x \to 0} \frac{\cos x - 1}{x^2}\right)$$, by the product theorem for limits

$$= 0 \cdot \left(\frac{-1}{2}\right) = 0.$$

(6)(a) Let $X$ be a set.

A *topology* on $X$ is a collection $\mathcal{T}$ of subsets of $X$ such that

(a) $\phi \in \mathcal{T}$ and $X \in \mathcal{T}$

(b) If $\mathcal{U} \subseteq \mathcal{T}$ then $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$

(c) If $n \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_n \in \mathcal{T}$ then

$$U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{T}$$

(b) A *topological space* is a set $X$ with a topology $\mathcal{T}$ on $X$.

(c) Let $X$ be a topological space with topology $\mathcal{T}$.

An *open set* of $X$ is a subset $U \subseteq X$ such that $U \in \mathcal{T}$.

(d) A *closed set* of $X$ is a subset $K \subseteq X$ such that $K^c \in \mathcal{T}$, where $K^c = \{x \in X \mid x \notin K\}$.

(e) Let $X$ be a topological space and let $E \subseteq X$.

The *interior of $E$* is a subset $E^\circ$ such that

(a) $E^\circ \subseteq X$ and $E^\circ$ is open

(b) $E^\circ \subseteq E$

(c) If $U \subseteq X$ is open and $U \subseteq E$ then $U \subseteq E^\circ$.

(f) Let $X$ be a topological space and let $E \subseteq X$.

The <u>closure of $E$</u> is the set $\bar{E} \subseteq X$ such that

(a) $\bar{E}$ is closed,

(b) $\bar{E} \supseteq E$,

(c) if $K \subseteq X$ is closed and $K \supseteq E$ then

$K \supseteq \bar{E}$.

(g) Let $X$ be a topological space and let $E \subseteq X$.

An <u>interior point of $E$</u> is $x \in E$ such that there exists a neighborhood $N$ of $x$ with $N \subseteq E$.

(h) A <u>close point of $E$</u> is $x \in X$ such that if $N$ is a neighborhood of $x$ then $N \cap E \neq \emptyset$.

(i) Let $X$ be a <u>topological space</u> and let $x \in X$.

A <u>neighborhood of $x$</u> is a set $N \subseteq X$ such that there exists an open set $U$ with $x \in U \subseteq N$.

<u>Examples</u> Let $X = \mathbb{R}$ and let $\mathcal{T}$ be set of subsets of $\mathbb{R}$ which are unions of intervals of the form

$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$,

$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$, and

$(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$.

Let $\mathcal{T}$ is a <u>topology on $\mathbb{R}$</u> and $X = \mathbb{R}$ is a topological space.

The interval $(0,1)$ is open in $\mathbb{R}$
and the interval $[0,1]$ is closed in $\mathbb{R}$.

The closure of $(0,1)$ is $[0,1]$.

The interior of $[0,1]$ is $(0,1)$.

The closure of $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ is $\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$

The interior of $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ is $\phi$.

The interior points of $[0,1]$ are the elements of the set $(0,1)$

The point $0$, is a close point to $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

(7) Let $X$ and $Y$ be metric spaces with distance functions $d_X : X \times X \to \mathbb{R}_{\geq 0}$ and $d_Y : Y \times Y \to \mathbb{R}_{\geq 0}$.

Let $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$. The <u>$\varepsilon$-ball at $x$</u> is

$$B_\varepsilon(x) = \{p \in X \mid d_X(p, x) < \varepsilon\}$$

The topology on $X$ is given by setting

$E \subseteq X$ is <u>open</u> if $E$ is a union of $\varepsilon$-balls in $X$

Let $\varepsilon \in \mathbb{R}_{>0}$ and $y \in Y$. The <u>$\varepsilon$-ball at $y$</u> is

$$B_\varepsilon(y) = \{q \in Y \mid d_Y(q, y) < \varepsilon\}.$$

The topology on $Y$ is given by setting

$F \subseteq Y$ is <u>open</u> if $F$ is a union of $\varepsilon$-balls in $Y$.

Let $f : X \to Y$ be a function.

The function $f : X \to Y$ is <u>continuous</u> (as a function between metric spaces) if $f$ satisfies:

If $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $p \in X$ and $d_X(p, x) < \delta$ then $d_Y(f(p), f(x)) < \varepsilon$.

If $F \subseteq Y$ let

$$f^{-1}(F) = \{x \in X \mid f(x) \in F\}.$$

The function $f : X \to Y$ is <u>continuous</u> (as a function between topological spaces) if $f$ satisfies:

if $F \subseteq Y$ is open then $f^{-1}(F)$ is open in $X$.

(8) Let $a, b \in \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$.

Let $c \in [a,b]$. Then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Assume that $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are functions and that

$f'(c)$ and $g'(c)$ exist.

To show: $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.

$$LHS = (fg)'(c) = \lim_{x \to c} \frac{(fg)(x) - (fg)(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c},$$ by the definition of the function $fg$,

$$= \lim_{x \to c} \frac{(f(x) - f(c))g(x) + f(c)g(x) - f(c)g(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot g(x) + f(c) \cdot \frac{g(x) - g(c)}{x - c}$$

$$= \lim_{x \to c} \left( \frac{f(x) - f(c)}{x - c} \cdot g(x) \right) + \lim_{x \to c} \left( f(c) \left( \frac{g(x) - g(c)}{x - c} \right) \right),$$ by the sum theorem for limits

$$= \left( \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \right) \left( \lim_{x \to c} g(x) \right) + \left( \lim_{x \to c} f(c) \right) \left( \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \right),$$

by the product theorem for limits

$= f'(c) \lim_{x \to c} g(x) + f(c) g'(c)$, by the definition of $f'(c)$ and $g'(c)$

$= f'(c) g(c) + f(c) g'(c)$, since $\lim_{x \to c} \frac{g(x)-g(c)}{x-c}$ exists implies $\lim_{x \to c} g(x) = g(c)$.

So

$(fg)'(c) = f'(c) g(c) + f(c) g'(c)$.

(5a) (a) A <u>metric space</u> is a set $X$ with a function

$d: X\times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $p\in X$ then $d(p,p)=0$,

(b) If $p,q\in X$ and $p\neq q$ then $d(p,q)\neq 0$,

(c) If $p,q\in X$ then $d(p,q)=d(q,p)$.

(d) If $p,q,r\in X$ then $d(p,q)\leq d(p,r)+d(r,q)$.

<u>Examples</u>: (a) $X=\mathbb{R}$ with $d: X\times X\to \mathbb{R}_{\geq 0}$ given by

$d(p,q)=|q-p|$, is a metric space

(b) $X=\mathbb{R}^2$ with $d: X\times X\to\mathbb{R}_{\geq 0}$ given by

$d(p,q)=\sqrt{(p_1-q_1)^2+(p_2-q_2)}$ for $p=(p_1,p_2)$, $q=(q_1,q_2)$.

(b) Let $X$ and $Y$ be metric spaces and let $a\in X$ and $l$ in $Y$. The <u>limit of $f$ as $x$ approaches $a$ is $l$</u> if $f$ satisfies:

if $\varepsilon\in\mathbb{R}_{>0}$ then there exists $\delta\in\mathbb{R}_{>0}$ such that if ~~$d(x,a$~~ $x\in X$ and $d_X(x,a)<\delta$ then

$d_Y(f(x),l)<\varepsilon$,

where $d_X: X\times X\to\mathbb{R}_{\geq 0}$ and $d_Y: Y\times Y\to\mathbb{R}_{\geq 0}$ are the distance functions on $X$ and $Y$ respectively.

(c) Let $X$ be a metric space and let $(x_n)$ be a sequence in $X$. Let $l \in X$.

The <u>limit of $(x_n)$ as $n \to \infty$ is $l$</u> if $(x_n)$ satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(x_n, l) < \varepsilon$.

(d) Let $f: X \to Y$ be a function between metric spaces $X$ and $Y$. Let $a \in X$. The function $f$ <u>is continuous at $x = a$</u> if $f$ satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $d_X(x, a) < \delta$ then $d_Y(f(x), f(a)) < \varepsilon$,

where $d_X: X \times X \to \mathbb{R}_{\geq 0}$ and $d_Y: Y \times Y \to \mathbb{R}_{\geq 0}$ are the distance functions on $X$ and $Y$, respectively.

(e) Let $f: X \to Y$ be a function between metric spaces $X$ and $Y$ (with distance functions $d_X: X \times X \to \mathbb{R}_{\geq 0}$ and $d_Y: Y \times Y \to \mathbb{R}_{\geq 0}$ respectively).

The function $f$ is <u>continuous</u> if $f$ satisfies:

If $\varepsilon \in \mathbb{R}_{>0}$ and $a \in X$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $d_X(x, a) < \delta$ then $d_Y(f(x), f(a)) < \varepsilon$.

(f) Let $f: X \to Y$ be a function between metric spaces. The function $f: X \to Y$ is <u>uniformly continuous</u> if $f$ satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, a \in X$ and $d_X(x,a) < \delta$ then $d_Y(f(x), f(a)) < \varepsilon$.

(g) The function $f: X \to Y$ is <u>Lipschitz</u> if there exists $K \in \mathbb{R}_{>0}$ such that

if $x, a \in X$ then $K d_X(x,a) \geq d_Y(f(x), f(a))$.

(h) Let $X$ be a metric space with distance function $d: X \times X \to \mathbb{R}_{\geq 0}$. Let $a \in X$ and $\varepsilon \in \mathbb{R}_{>0}$. The <u>$\varepsilon$-ball at $a$</u> is

$$B_\varepsilon(a) = \{x \in X \mid d_X(x,a) < \varepsilon\}.$$

<u>Examples</u> If $X = \mathbb{R}$ then

$$(a,b) = B_{\frac{b-a}{2}}\left(\frac{b+a}{2}\right) \quad \text{and} \quad (2,3) = B_{\frac{1}{2}}(2.5)$$

The function $f: (0,2) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous